# Week 1–3: Mathematics – Limit and Calculus

Kyeong Soo (Joseph) Kim

XJTLU

# Outline

- Speed and Velocity
- Limit
- Abstraction
- Calculus

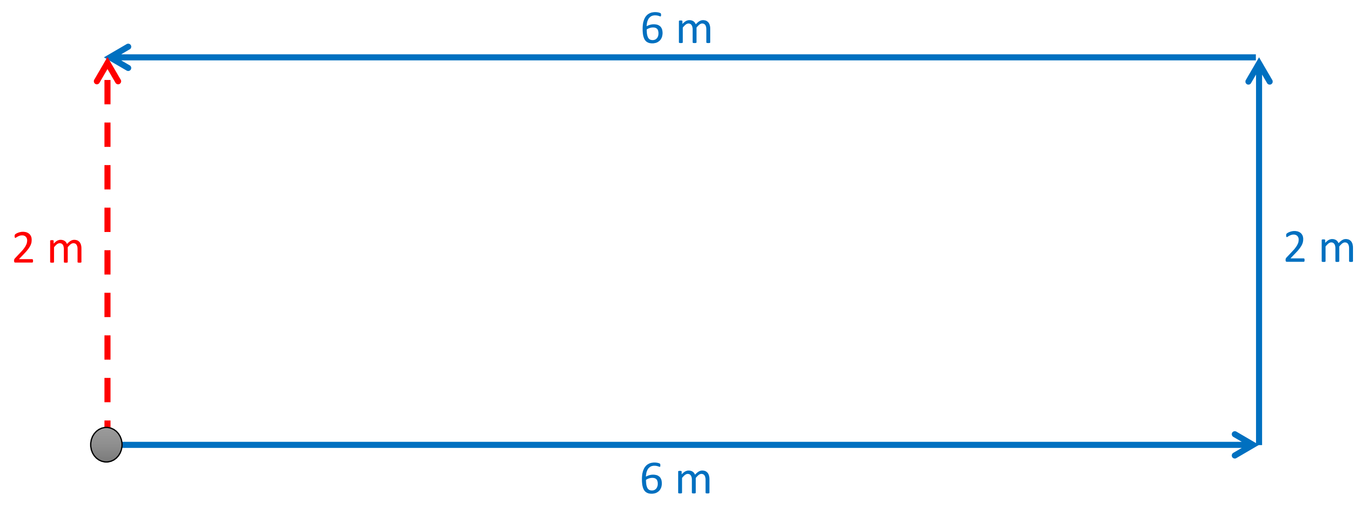
# Speed and Velocity

# Usain Bolt

# TGV

# Speed/속력 (速力) vs. Velocity/속도 (速度)

- Consider the blue movement over 10 second shown below.
  - Average speed (distance/time): (6 m + 2 m + 6 m)/10 s = 1.4 m/s
  - Average velocity (displacement/time): 2 m / 10 s = 0.2 m/s

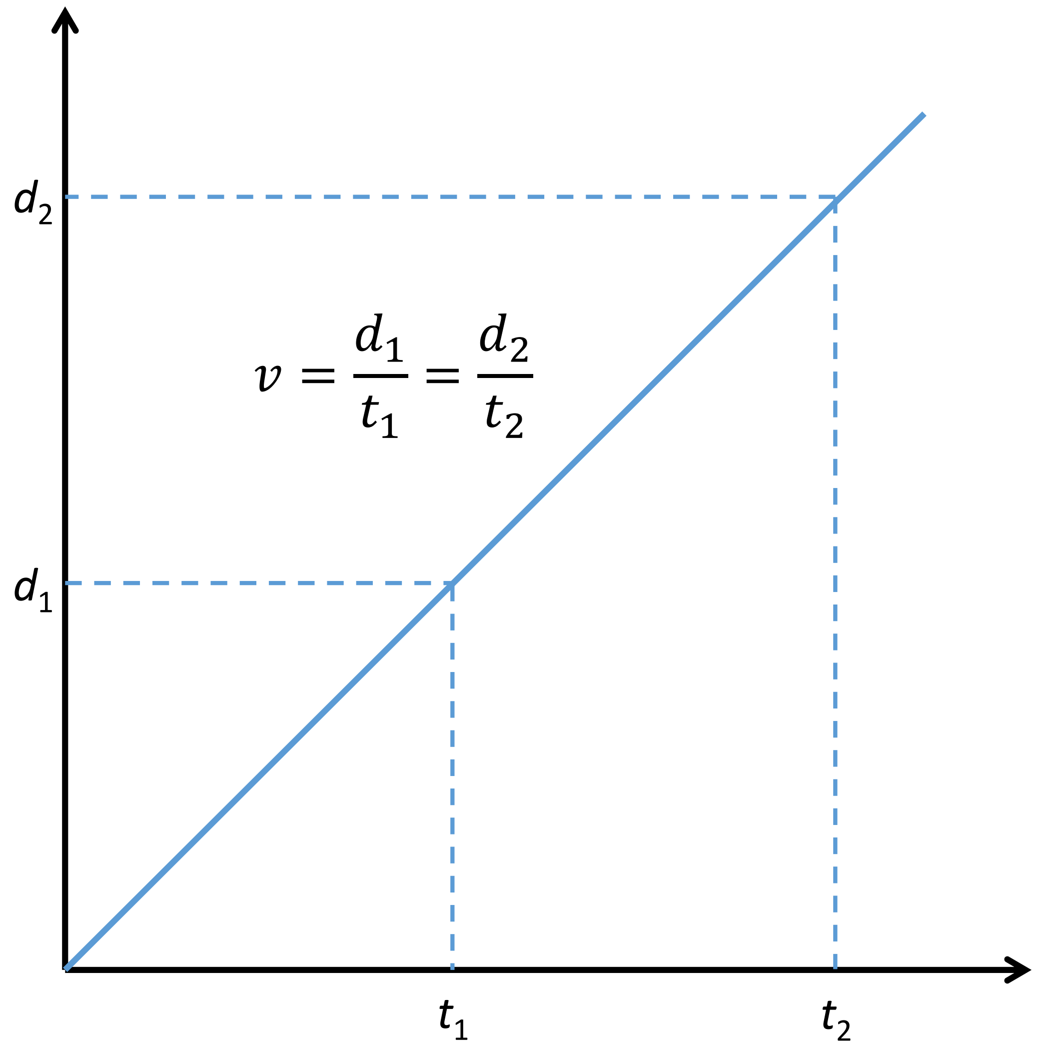


# Speed of Usain Bolt and TGV

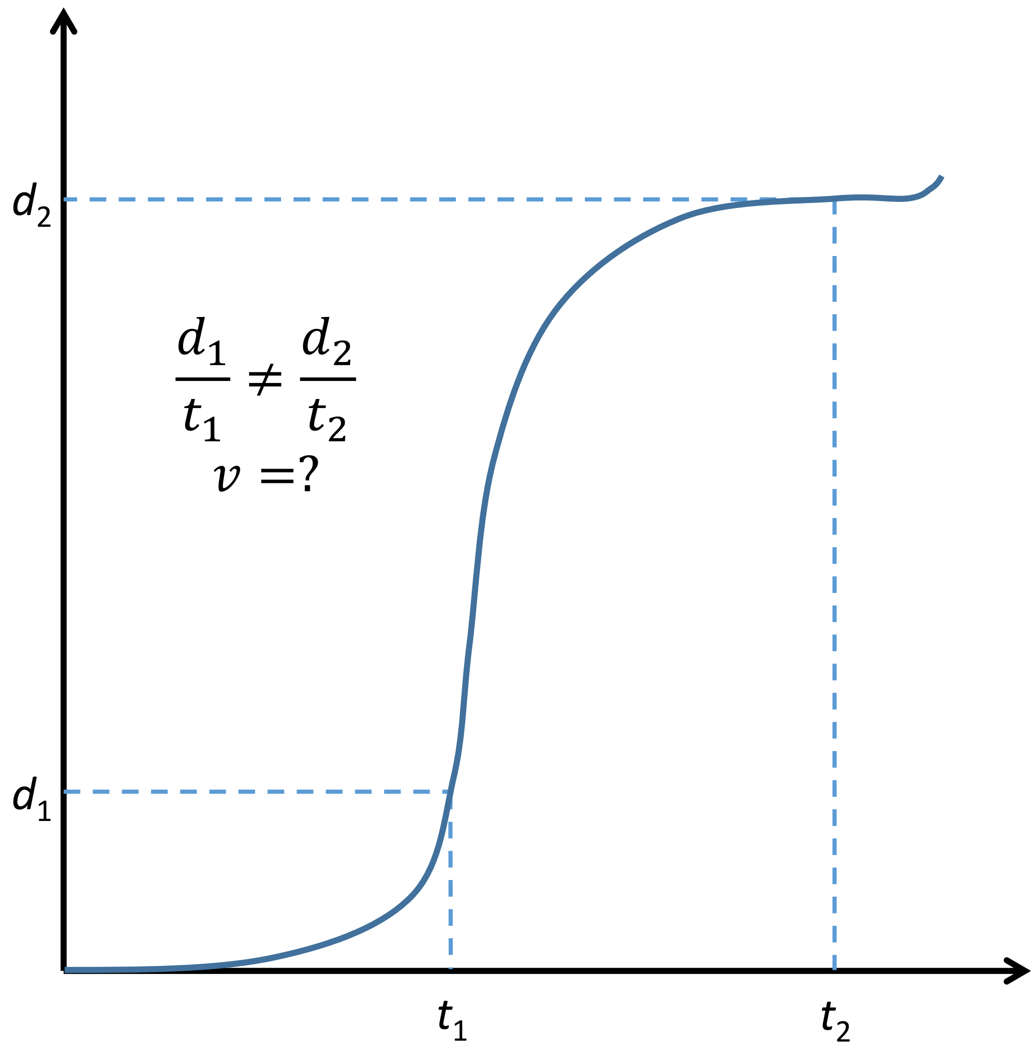
- For Usain Bolt
  - Distance (*d*): 100 m
  - Time it takes (*t*): 9.58 $s$
  - Speed (*v*): $v = \frac{d}{t} = \frac{100\ m}{9.58\ s} \approx 10.44\ {}^{m}/_{s} \approx 37.58\ {}^{km}/_{h}$

- For TGV
  - Distance (*d*): ?
  - Time it takes (*t*): ?
  - Speed (*v*): ?

# Average Speed



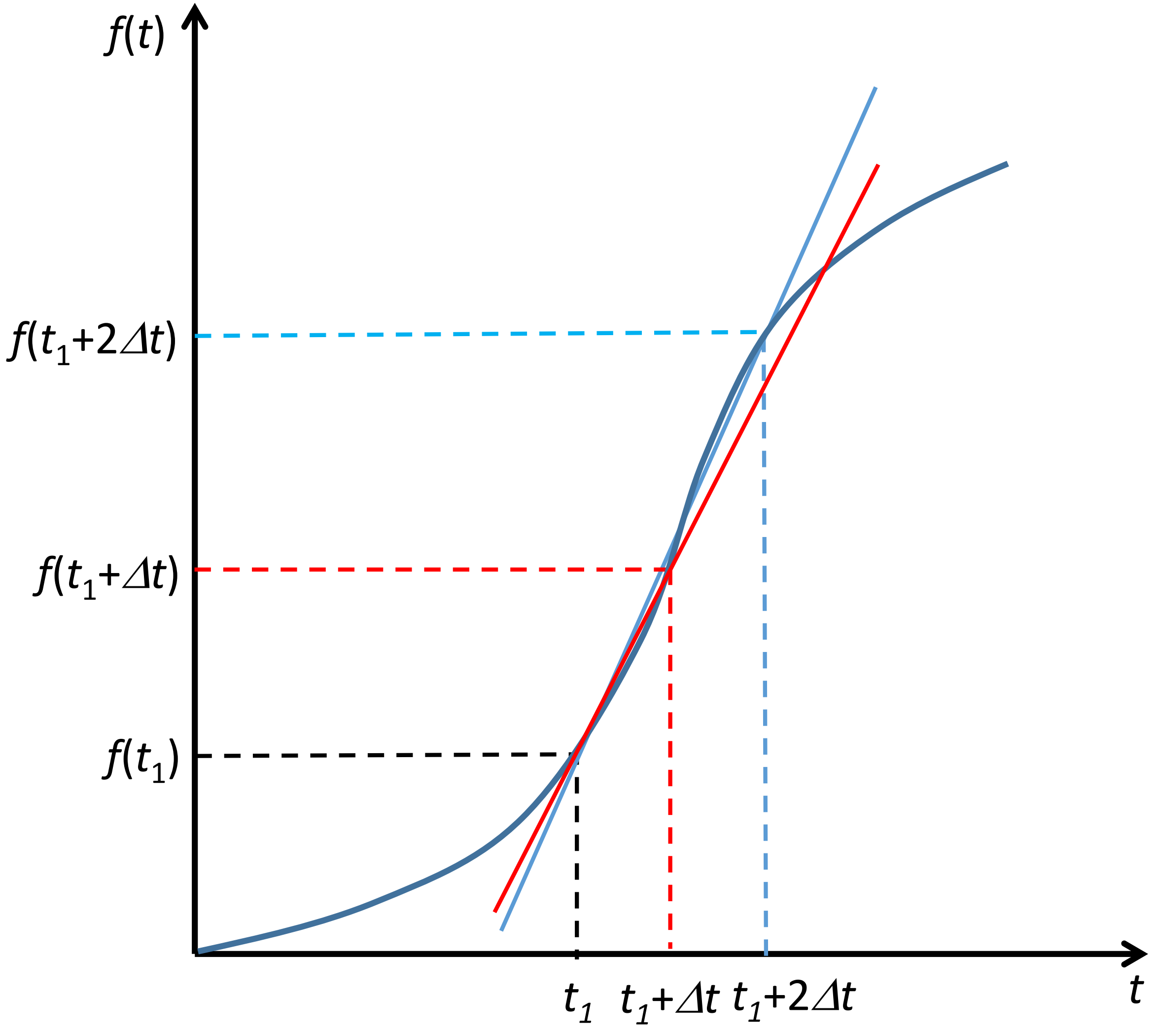

**vs.**

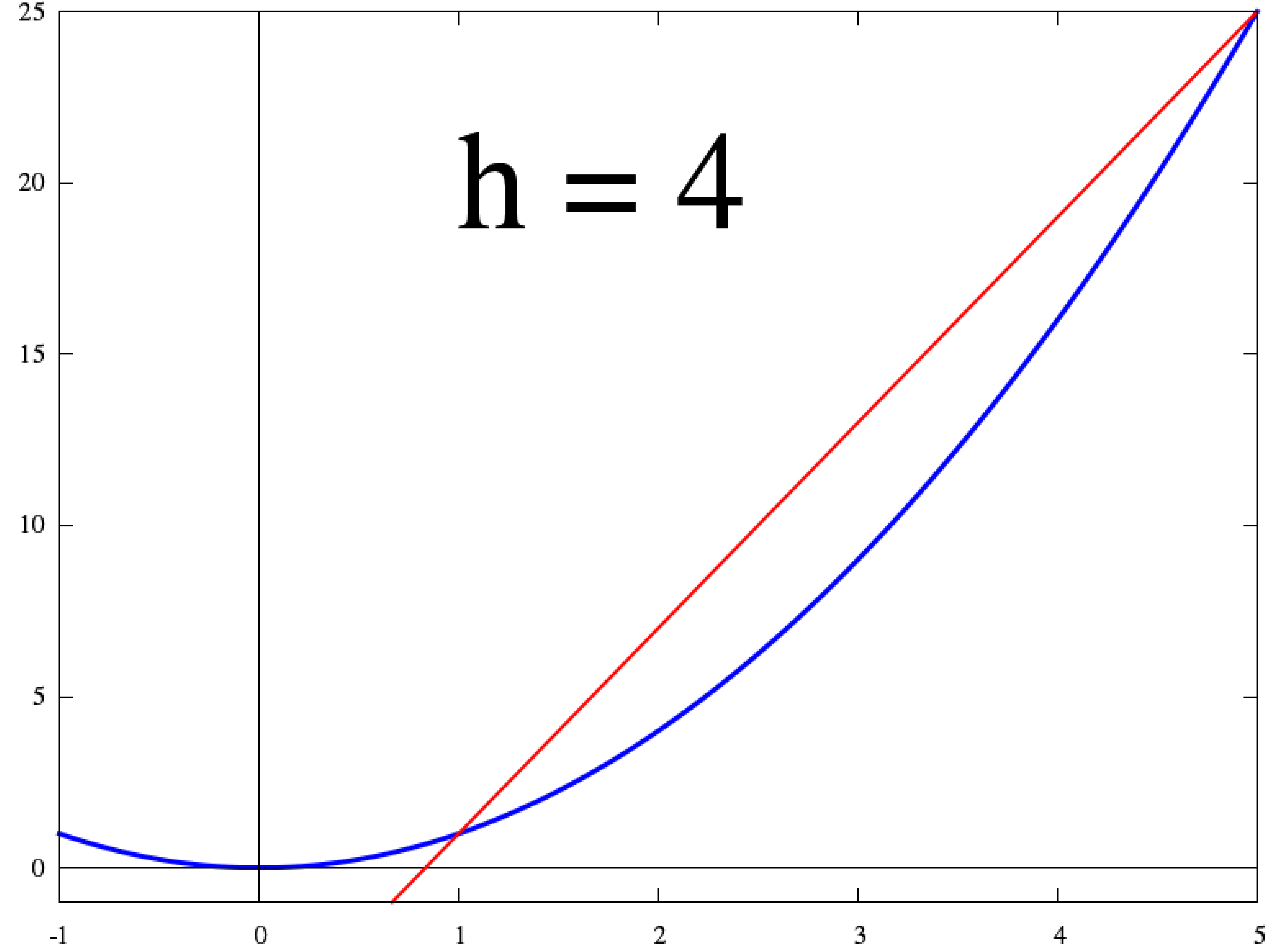


# Instantaneous Speed – Origin of Derivative

- Consider the average speed over different time intervals as follows:
  - $\frac{f(t+2\Delta t)-f(t)}{2\Delta t}$ for $[t, t+2\Delta t]$
  - $\frac{f(t+\Delta t)-f(t)}{\Delta t}$ for $[t, t+\Delta t]$

- What if $\Delta t$ becomes extremely small?
  - $f'(t) \triangleq \lim_{\Delta t \to 0} \frac{f(t+\Delta t)-f(t)}{\Delta t}$

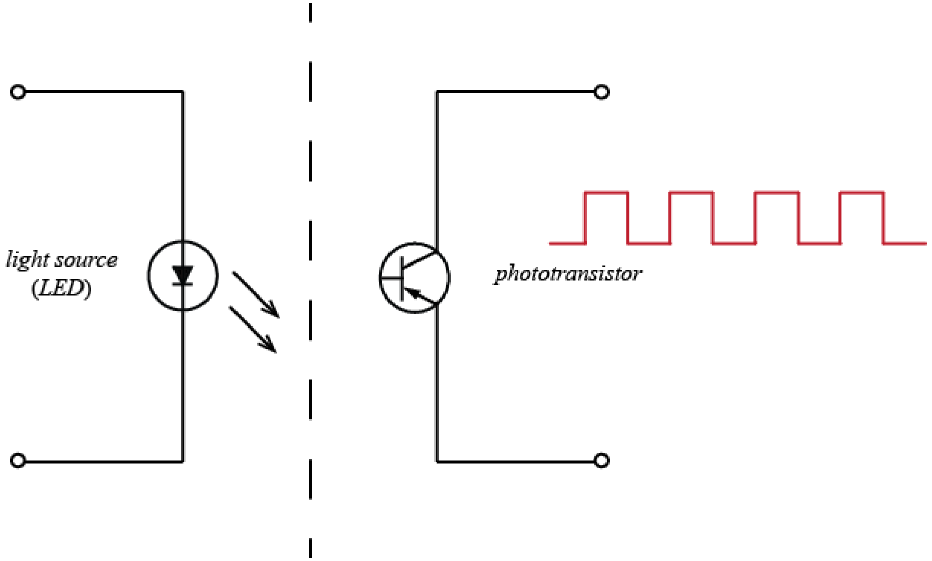




h = 4
25
20
15
10
5
0
-1
0
1
2
3
4
5

# How to Measure the Speed of Car/Train/…?

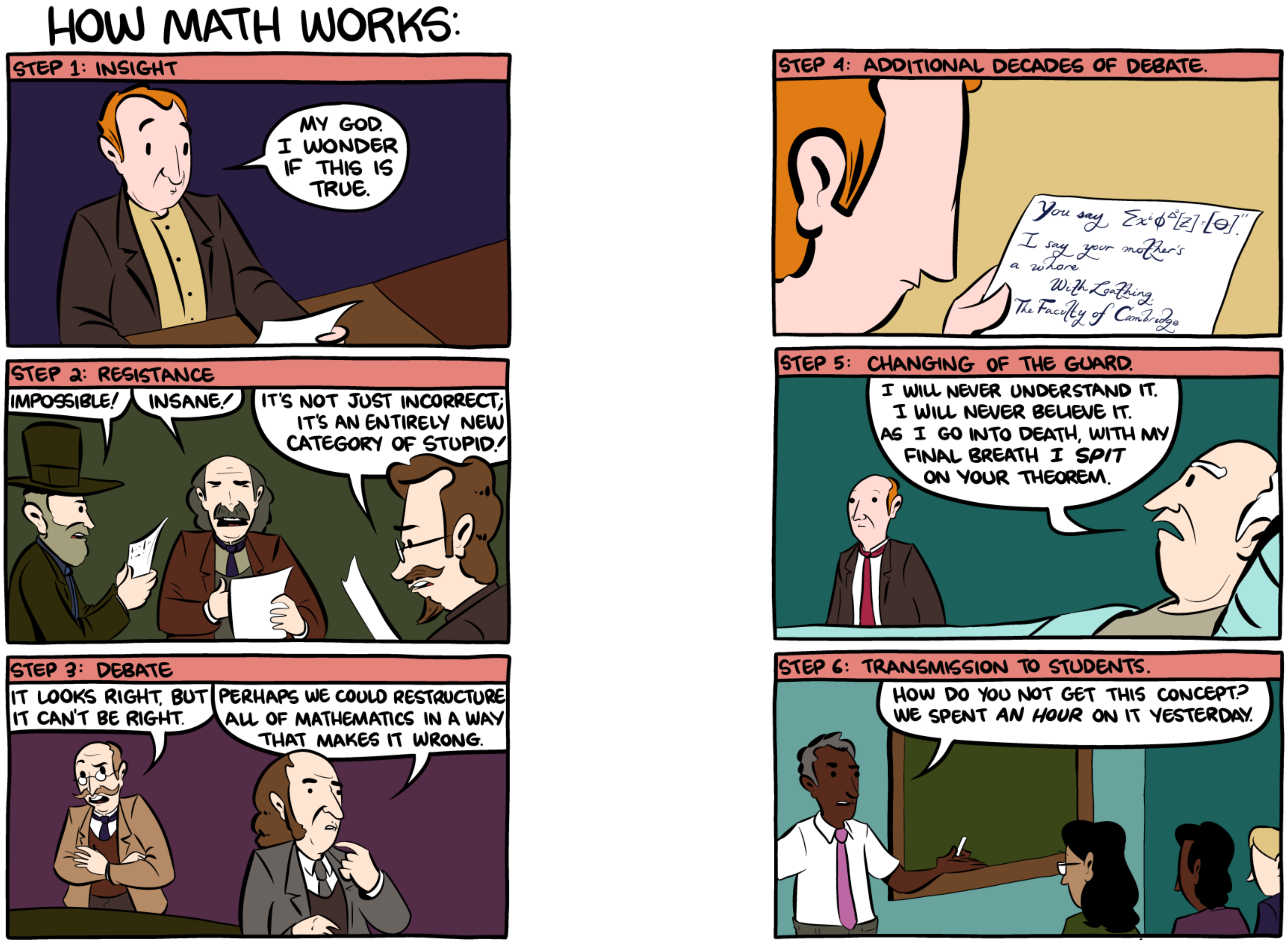
- Count the number of pulses per time unit.
  - e.g., 10 pulses/s
- Divide it by the number of slits in the disk.
  - e.g., 10 pulses/s ÷ 20 slits/rotation = 0.5 rotation/s
- Multiply it the circumference of a tire/wheel.
  - e.g., 0.5 rotation/s × 1 m/rotation = 0.5 m/s



# Abstraction

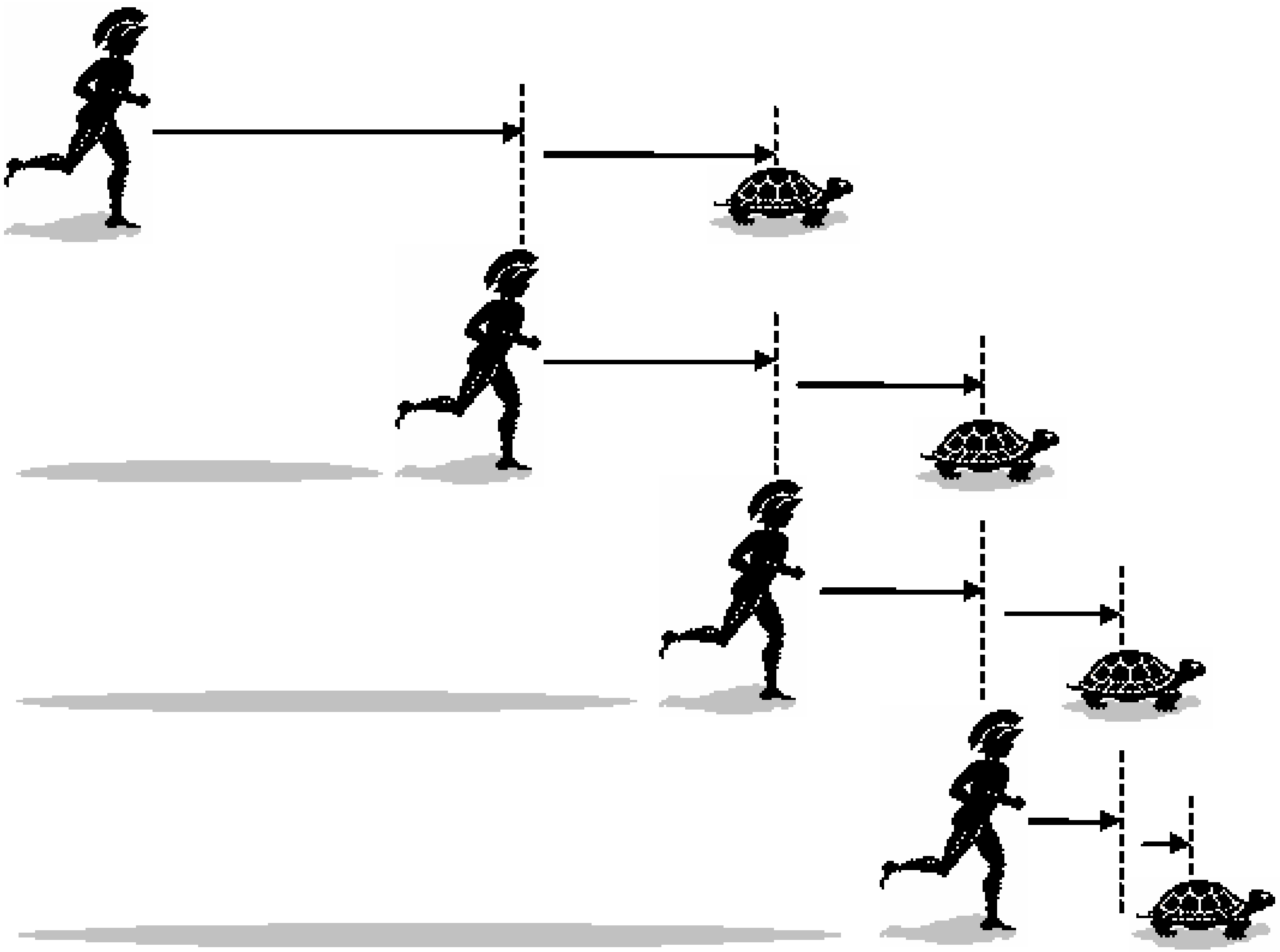
# What is abstraction/추상 (抽象)?

- Have you seen a dog?
- How about 0, 1, 2 …?


HOW MATH WORKS:
STEP 1: INSIGHT
MY GOD. I WONDER IF THIS IS TRUE.
STEP 2: RESISTANCE
IMPOSSIBLE!
INSANE!
IT'S NOT JUST INCORRECT; IT'S AN ENTIRELY NEW CATEGORY OF STUPID!
STEP 3: DEBATE
IT LOOKS RIGHT, BUT IT CAN'T BE RIGHT.
PERHAPS WE COULD RESTRUCTURE ALL OF MATHEMATICS IN A WAY THAT MAKES IT WRONG.
STEP 4: ADDITIONAL DECADES OF DEBATE.
You say "$\Sigma x^i \phi^\Delta[z] = [\Theta]$."
I say your mother's a whore.
With Loathing,
The Faculty of Cambridge
STEP 5: CHANGING OF THE GUARD.
I WILL NEVER UNDERSTAND IT. I WILL NEVER BELIEVE IT. AS I GO INTO DEATH, WITH MY FINAL BREATH I SPIT ON YOUR THEOREM.
STEP 6: TRANSMISSION TO STUDENTS.
HOW DO YOU NOT GET THIS CONCEPT? WE SPENT AN HOUR ON IT YESTERDAY.

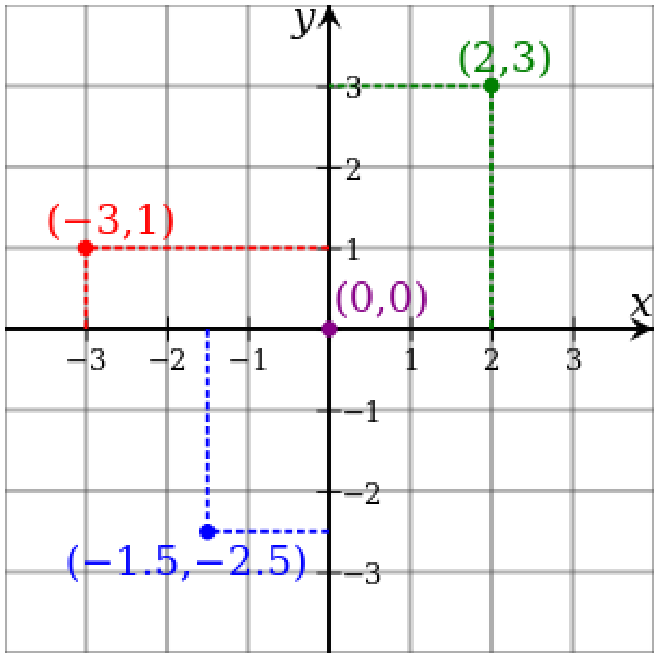
# Limit

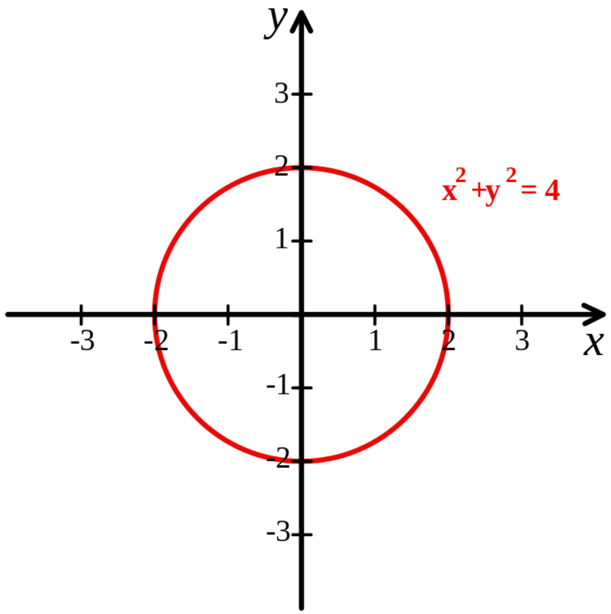
# Zeno’s Paradoxes - Achilles and the Tortoise



# Cartesian coordinate system*

- By René Descartes (1596-1650)
  - French Philosopher, Mathematician, and Scientist



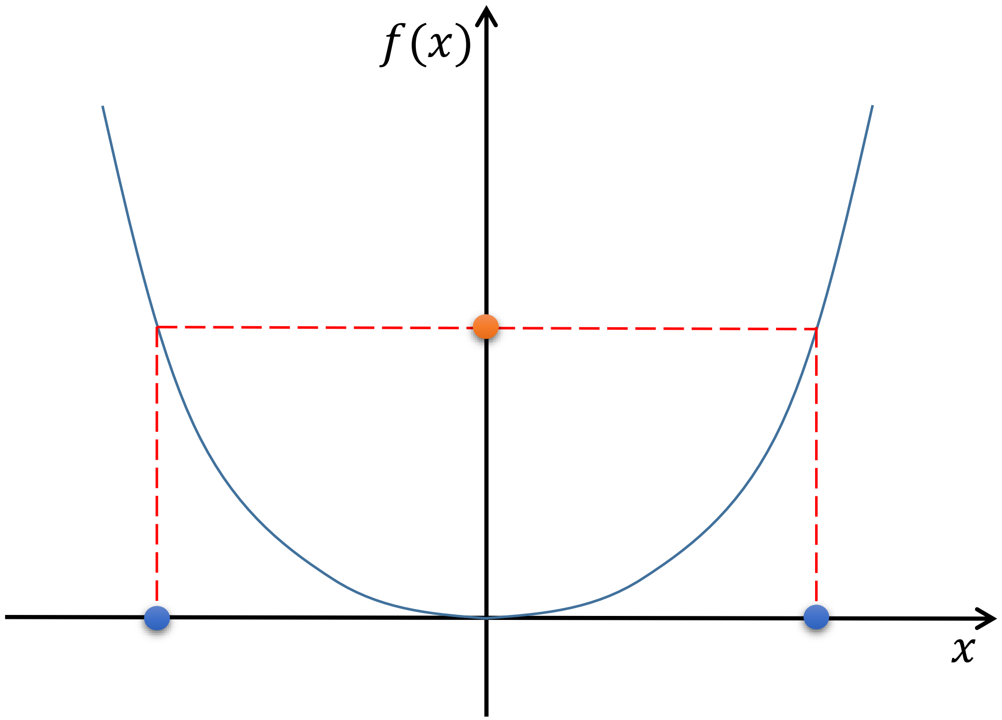




***I think, therefore I am.***

**** Based on Wikipedia.***

# Function

- A relation between a set of inputs and a set of permissible outputs with the property that each input is related to *exactly one output*.

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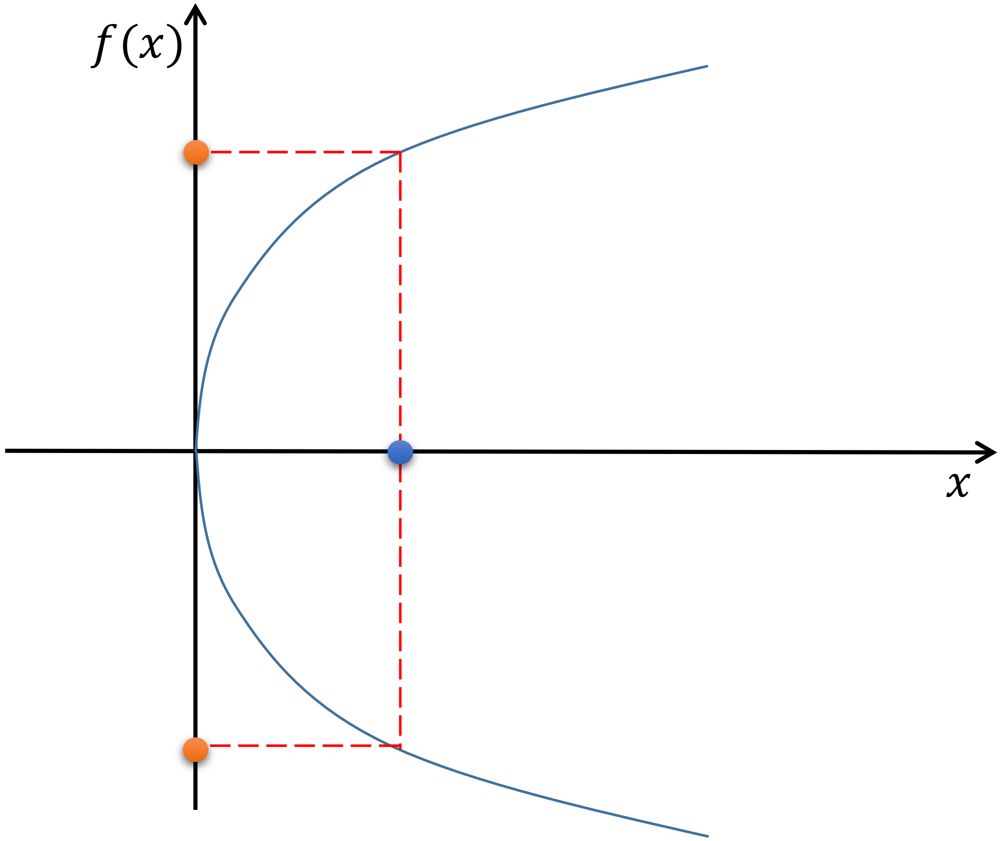
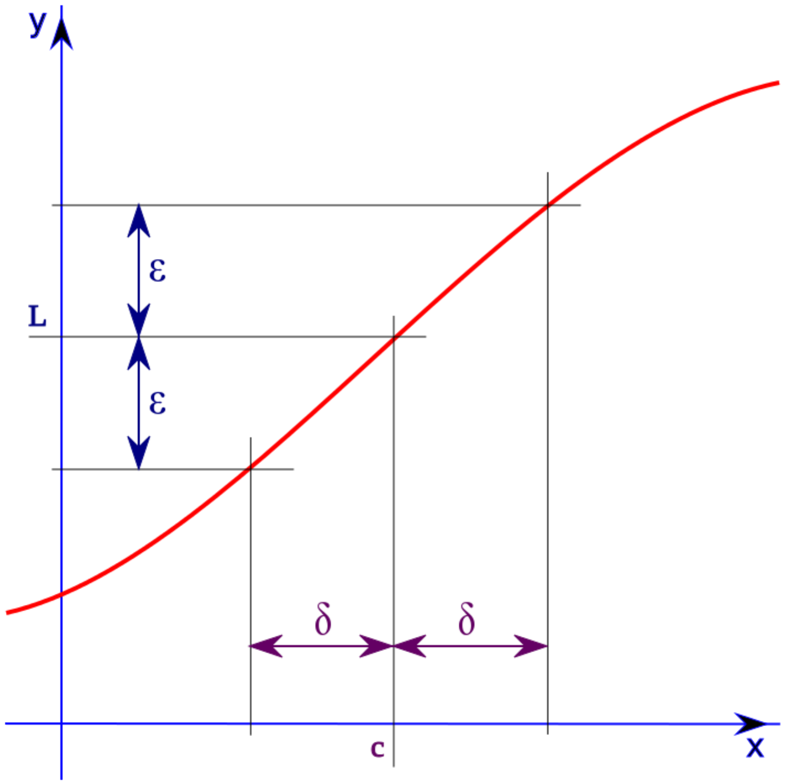


# $(\varepsilon, \delta)$-Definition of Limit

- Consider the following:
  - $f$: A real-valued function defined on a subset $D$ of the real numbers.
  - $c$: A limit point of $D$.
  - $L$: A real number.
- We say that

$$\lim_{x \to c} f(x) = L$$

if for every $\varepsilon > 0$ there exists a $\delta$ such that, for all $x \in D$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.



# $(\varepsilon, \delta)$-Definition of Limit - Example

- Let us prove

$$\lim_{x \to 5}(3x - 3) = 12.$$

- The key is to show how $\delta$ and $\varepsilon$ must be related to each other. Specifically, we want show that

$$0 < |x - 5| < \delta \Longrightarrow |(3x - 3) - 12| < \varepsilon.$$

- Simplifying, factoring and dividing 3 on the right hand side gives us

$$|x - 5| < \frac{\varepsilon}{3} \Longrightarrow \delta = \frac{\varepsilon}{3}\,.$$

# Calculus

# Differential Calculus

- Given a function $y = f(x)$, its *derivative* – written as $\frac{dy}{dx}$ – is a measure of *the rate* at which the value $y$ *of the function changes* with respect to the change of variable $x$.

- The derivative of the function $f$ at $a$ is defined as the limit, i.e.,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

- We can consider the derivative of the function $y = f(x)$ as another function that sends the point $x$ to the derivative $f$ at $x$, which is denoted as

$$f'(x) = \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

# Differential Calculus – Example 1

- Differentiate $x^2$.

$$\frac{d}{dx}x^2 = \lim_{h\to 0}\frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0}\frac{(x^2 + 2xh + h^2) - x^2}{h}$$

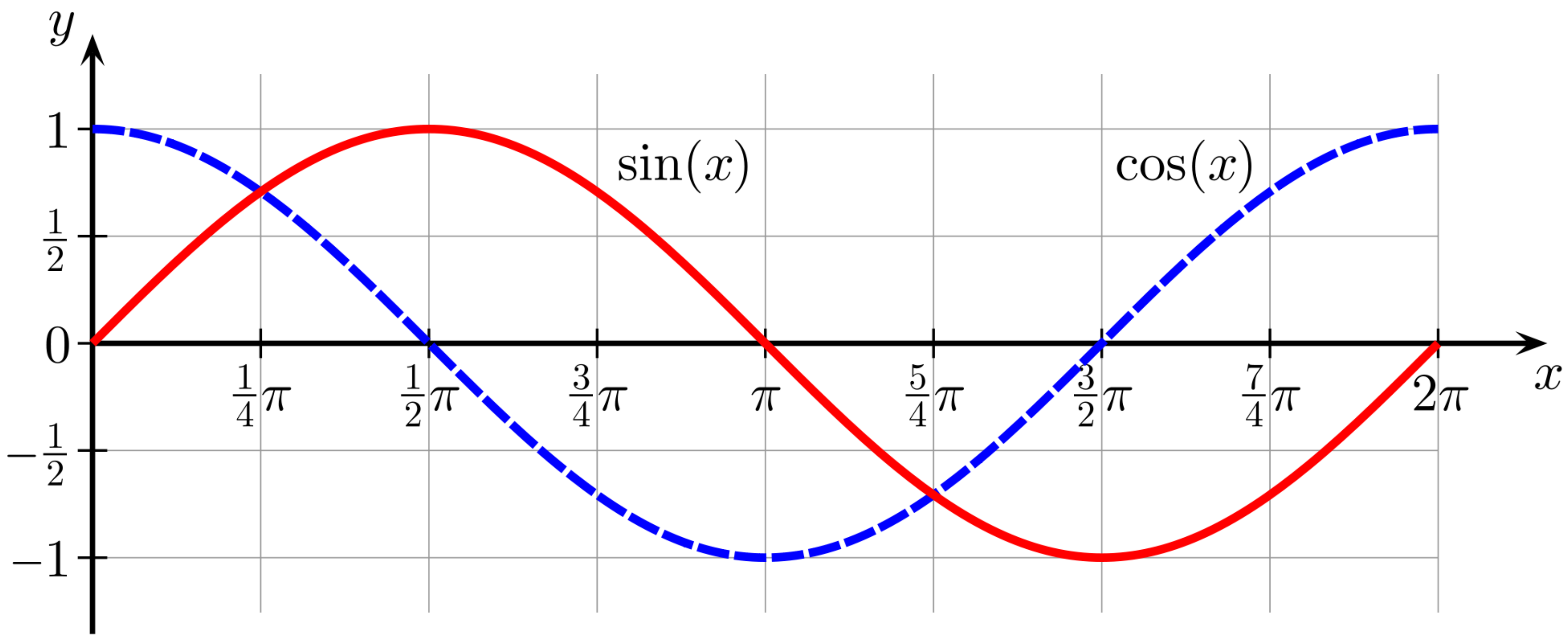
$$= \lim_{h\to 0}\frac{h(2x+h)}{h}$$

$$= \lim_{h\to 0}\frac{h(2x+h)}{h}$$

$$= \lim_{h\to 0}(2x+h) = 2x + \lim_{h\to 0} h = 2x$$

# Differential Calculus – Example 2

- $\frac{d}{dx}\sin x = \cos x$
- $\frac{d}{dx}\cos x = -\sin x$

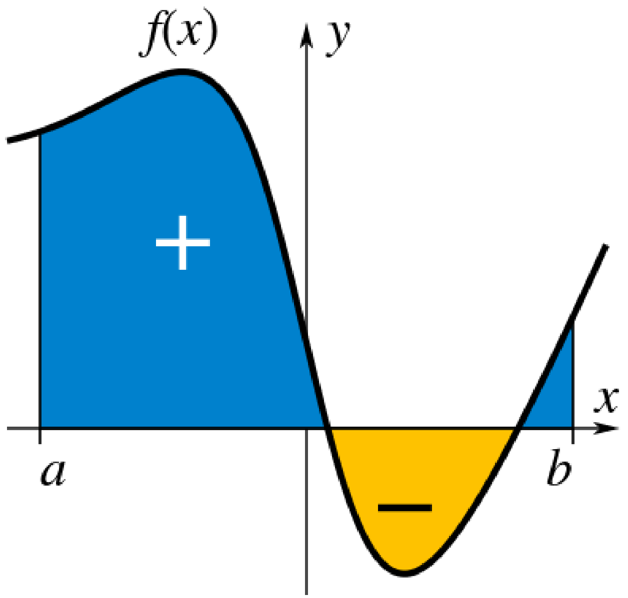


# Integral Calculus

- Given a function $y = f(x)$ and interval [a, b] of the real line, the *definite integral*

$$\int_a^b f(x)dx$$

  is defined as the *signed area* of the region bounded by the graph of $f(x)$, the x axis and two vertical lines $x = a$ and $x = b$.



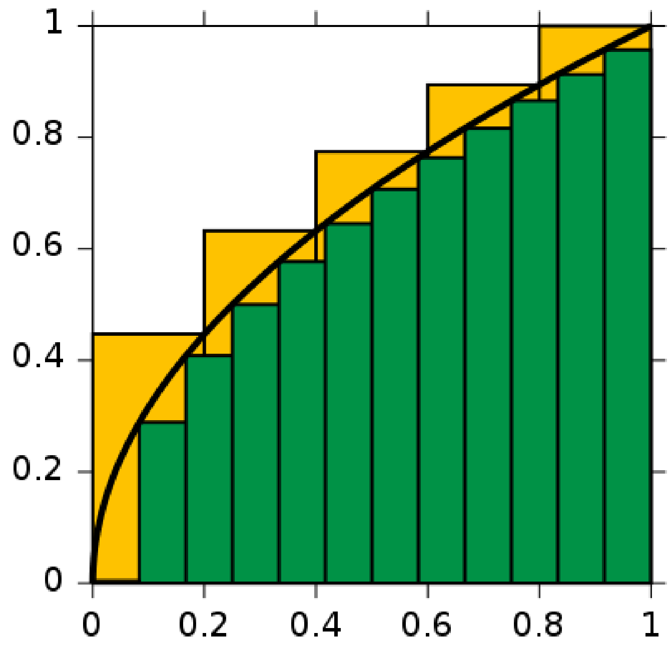

- The reverse of differentiation is defined as an *indefinite integral*, i.e.,

$$F(x) = \int f(x)$$

  - *Note that there is no interval.*

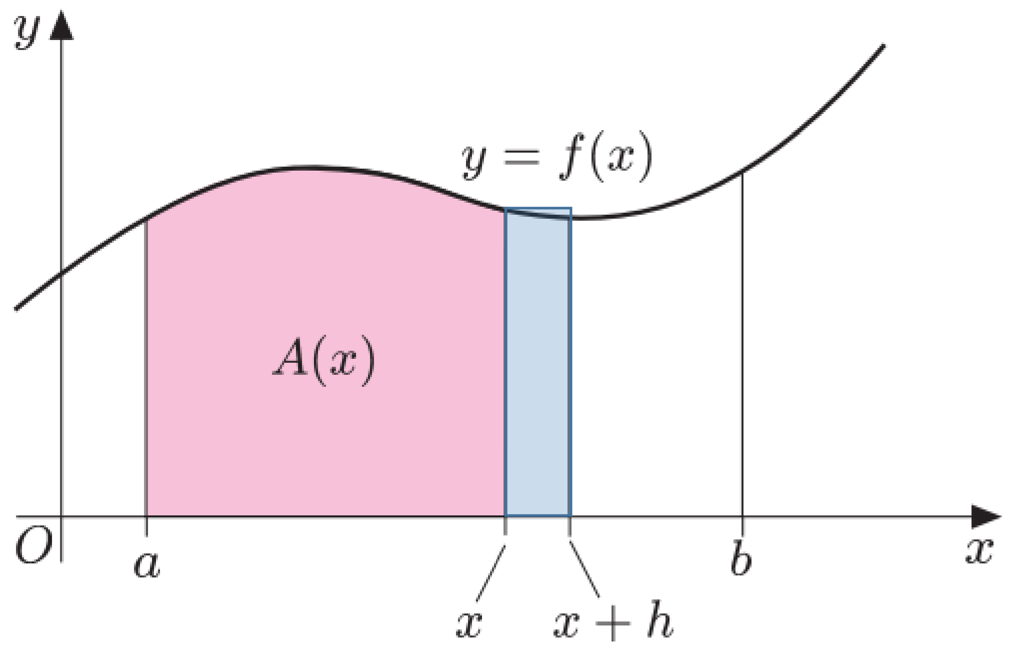
- With the indefinite integral, we can express the definite integral as

$$\int_a^b f(x)dx = F(b) - F(a)$$



# Relationship between Differentiation and Integration

$$\lim_{h \to 0} \frac{A(x+h) - A(x)}{h} = ?$$




- For a very small $h$, the area under the curve $f(x)$ from $x$ to $x + h$ can be approximated as a rectangle, i.e.,
  - $A(x+h) - A(x) \approx hf(x)$
- Dividing it by $h$ and taking limit, we obtain
  - $\lim_{h \to 0} \frac{A(x+h)-A(x)}{h} = f(x)$
  - i.e., $\frac{d}{dx}A(x) = f(x)$